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# A Burgers concentration dispersion equation

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In a non-vertical borehole light particles tend to rise towards the upper side of the borehole. The resulting non-uniform density distribution tends to induce an upwards contribution to the longitudinal flow along that upper side of the flow, with a compensating downflow elsewhere. On average the particles experience an extra upflow proportional to the cross-sectionally averaged concentration of particles. Mathematically this concentration-related change of speed corresponds to the nonlinearity of the Burgers equation. Such is the strength of the buoyancy effect that in realistic flow conditions the Burgers nonlinearity can be significant for particle volume fractions of only one part per thousand.

## 1. Introduction

Two-phase flows are ubiquitous in the oil drilling industry. In the early stages of drilling the flow of fluid conveys particles of crushed rock. At later stages there can be gas bubbles or oil droplets. It is almost inevitable that there will be density differences between the drilling fluid and the particles, bubbles or droplets. So, the flow can be changed by the presence of the conveyed material and the conveyed material need not travel at the same rate as the drilling fluid. As gases rise from several kilometers deep, they expand by factors of several hundreds. The devastating effect of blowouts stems from the uncontrolled escalation of the upwards buoyancy forces. This paper focuses attention upon the first, most gentle, effects of buoyancy upon the distribution of dilute particles or bubbles along a non-vertical borehole.

For a non-buoyant solute in a parallel flow Taylor (1953) drew attention to the way that the non-uniform velocity distorts and stretches a region of marked fluid. He showed that the eventual longitudinal evolution of the cross-sectionally averaged concentration $\bar{c}(z,t)$ is diffusive:

$$\partial_t \bar{c} + w_0 \partial_z \bar{c} - (\kappa + D_0)\partial_z^2 \bar{c} = 0. \tag{1.1}$$

The shear dispersion coefficient $D_0$ depends both upon mixing across the flow and upon the non-uniformity (shear) of the velocity profile $w(x,y)$. Usually the molecular (or turbulent) diffusivity $\kappa$ is negligible relative to $D_0$. For dilute suspensions of rising or sinking particles Giddings (1968) showed that the same type of model equation remains valid. However, the effective longitudinal velocity $w_0$ and the shear dispersion coefficient $D_0$ involve particle-weighted averaged across the flow, so are modified by longitudinal and transverse drift.

The classical paper on buoyancy effects was written by Erdogan & Chatwin (1967). They showed that for solutes the longitudinal velocity $w_0$ remained unchanged. However, there is a transverse (or secondary) flow proportional to the density gradient, which augments mixing across the flow and results in a nonlinear shear dispersion coefficient:

$$\partial_t \bar{c} + w_0 \partial_z \bar{c} - \partial_z[\bar{\kappa} + D_0 + D_1 \partial_z \bar{c} + D_2(\partial_z \bar{c})^2 + \ldots] \partial_z \bar{c} = 0. \tag{1.2}$$

For horizontal flows the odd coefficients $D_1, \ldots$ are zero (Erdogan & Chatwin 1967).

In the present paper we find that for a dilute suspension of rising particles in an inclined flow the first, most gentle, effects of buoyancy are evidenced in a perturbed longitudinal velocity:

$$\partial_t \bar{c} + [w_0 + w_1 \bar{c}] \partial_z \bar{c} - [\bar{\kappa} + D_0] \partial_z^2 \bar{c} = 0. \tag{1.3}$$

Light particles on the upper side of the flow tend to give an upwards contribution to the longitudinal flow experienced by the particles. Physically this is related to the mechanism that causes suspended material to settle out most rapidly in a tilted container (Herbolzheimer & Acrivos 1981). In practice the velocity perturbation $w_1 \bar{c}$ can be substantial for particle volume fractions of only one part per thousand. For solutes or for non-tilted flows there is no asymmetric buoyancy effect and the nonlinear perturbation coefficient $w_1$ is zero. Equation (1.3) is a Burgers (1948) equation. To obtain solutions for the longitudinal concentration $\bar{c}(z,t)$ we can take advantage of the elegant solution technique developed by Hopf (1950) and Cole (1951).

## 2. Flow and mixing in a narrow gap

The geometry of boreholes is typically annular (see figure 1). There is an off-centre cylindrical shaft which contains the downflow of drilling fluid. The return flow, conveying bubbles or particles, occupies the outer annulus. We denote the mean gap width by $H$ and the mean gap radius by $a$. In practice the ratio $H/a$ is about 0.4.

In view of the fact that the ratio $H/a$ is not particularly small, many analyses pertaining to boreholes contend with the full complexity of the off-centre eccentric annular geometry (Snyder & Goldstein 1965; Sankarasubramanian & Gill 1971; Muller & Bittleston 1992). However, the errors incurred by making a narrow-gap approximation (and regarding the flow as being locally planar) are of order $(H/a)^2$. Indeed, Smith (1990*b*, equations (8.10), (8.11)) shows that for solute dispersion the narrow-gap approximation yields a shear dispersion coefficient $D_0$ within 10% of the exact values computed by Sankarasubramanian & Gill (1971) for three different off-centre geometries all with $H/a = 0.4$. Accordingly, the present work takes advantage of the narrow-gap approximation.

If we make the Boussinesq approximation (that for dilute suspensions changes in mass of the fluid are negligible but that changes in weight are important), then the equation for the conservation of mass can be averaged across the narrow gap to give

$$\partial_t h + \frac{1}{a} \partial_\theta (hv) + \partial_z (hw) = 0. \tag{2.1}$$

Here $h(\theta, z, t)$ is the gap width, ($\theta$ the angle around the shaft measured from its horizontal diameter), $v(\theta, z, t)$ the velocity component around the shaft, and $w(\theta, z, t)$ the axial velocity. The illustrative examples given in this paper concern the gap geometries (see figure 2)

$$h(\theta; \epsilon, \Theta) = H(1 - \epsilon \cos(\theta - \Theta)), \tag{2.2}$$

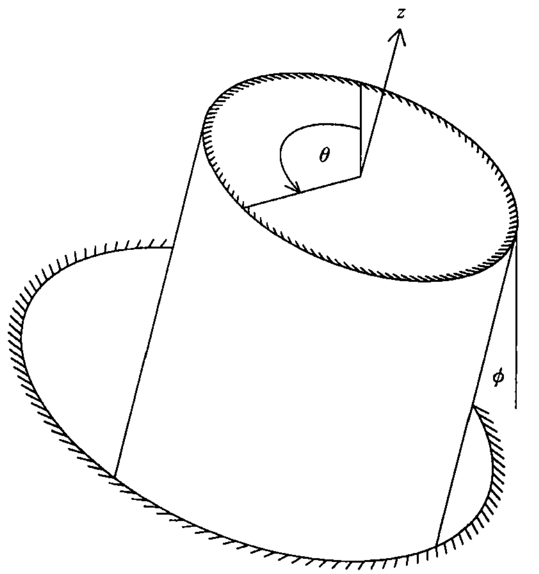


FIGURE 1. Sketch showing an off-centre drilling shaft in a non-vertical borehole. The line $\theta = 0$ is horizontal and the angle $\phi$ measures the deviation from vertical of the $z$-axis.

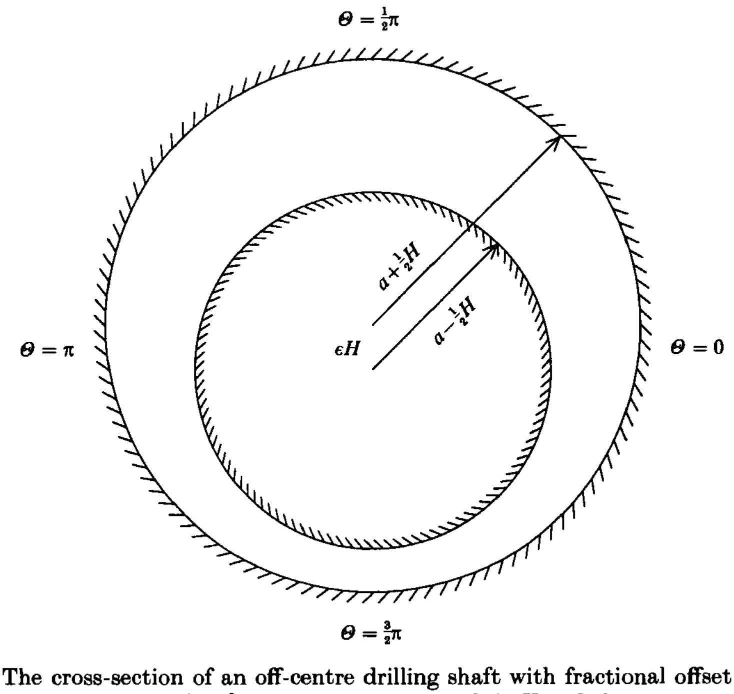


FIGURE 2. The cross-section of an off-centre drilling shaft with fractional offset $\epsilon = 0.5$ and narrowest gap at the angle $\Theta = \frac{3}{2}\pi$. The mean gap width is $H$ and the mean gap radius is $a$.

where $\epsilon$ is the fractional offset and $\Theta$ is the position of minimum gap. In figures 1, 2, 5 and 7 the minimum gap is at the bottom, $\Theta = \frac{3}{2}\pi$, and the fractional offset has the value $\epsilon = 0.5$.

For ease of exposition the flow will be calculated as if it were laminar. Smith (1990*a*) shows in detail how the corresponding calculations for solutes can be adapted to turbulent flows. When averaged cross the narrow gap, the momentum equations for laminar flow take the lubrication theory form (Schlichting 1955, section 6):

$$\frac{1}{a\rho_0}\partial_\theta p = -12\frac{\nu}{h^2}(v-\tfrac{1}{2}\Omega a)-\frac{\rho' g}{\rho_0}\cos\theta\sin\phi, \tag{2.3a}$$

$$\frac{1}{\rho_0}\partial_z p = -12\frac{\nu}{h^2}w-\frac{\rho' g}{\rho_0}\cos\phi. \tag{2.3b}$$

Here $\rho_0$ is the reference density of the drilling fluid, $p(\theta,z,t)$ the excess pressure (above reference hydrostatic), $\nu$ the kinematic viscosity, $\Omega$ the angular velocity of the drilling shaft, $\rho'(\theta,z,t)$ the density perturbation, $g$ the gravitational acceleration, and $\phi$ is the angle between the borehole axis and vertical. The effect of shaft rotation has been investigated by Smith (1990*b*). So, to avoid undue duplication, attention in the present paper is restricted to a stationary shaft ($\Omega = 0$).

We shall assume that the (very small) density perturbation $\rho'$ is directly proportional to the volume fraction $c(\theta,z,t)$ of particles (or bubbles) in the flow:

$$\rho' = -\alpha c\rho_0, \tag{2.4}$$

where $\alpha$ is positive for a buoyant substance. On the hypothesis that the suspension of particles (or bubbles) is very dilute, we shall ignore any corresponding (very small) perturbations in the viscosity or diffusivity.

The narrow-gap equation for $c(\theta,z,t)$ differs from that for solutes in the presence of axial and transverse components of the vertical rise velocity $\mathscr{W}$:

$$\partial_t(hc)+\frac{1}{a}\partial_\theta(h[v+V\cos\theta]c)+\partial_z(h[w+W]c) = \frac{1}{a^2}\partial_\theta(h\kappa_2\partial_\theta c)+\partial_z(h[\kappa_3+D_{zz}]\partial_z c), \tag{2.5a}$$

with

$$V = \mathscr{W}\sin\phi, \quad W = \mathscr{W}\cos\phi. \tag{2.5b, c}$$

Here $\kappa_2$ and $\kappa_3$ are the effective circumferential and axial diffusivities for the particles (or bubbles) and $D_{zz}(\theta)$ is the local longitudinal shear dispersion coefficient associated with velocity differences across the gap. For unidirectional laminar flow Bugliarello & Jackson (1964) derived the result

$$D_{zz} = \frac{h^2w^2}{210\kappa_1}, \quad \text{where} \quad \kappa_1 = \kappa_2 = \kappa_3. \tag{2.6}$$

For turbulent flows the eddy diffusivity $\kappa_1$ is proportional to $hw$ (with coefficient of about 0.006). So, the shear dispersion coefficient $D_{zz}$ is likewise proportional to $hw$ but with a much larger coefficient (Elder 1959). Non-uniformity around the gap of the longitudinal velocity $w$ gives rise to a longitudinal shear dispersion coefficient $E$ that can greatly exceed $D_{zz}$ (Sankarasubramanian & Gill 1979).

As a consequence of buoyancy (or rotation of the drilling shaft) the flow is not unidirectional. There should be additional transverse $D_{\theta\theta}$ and skew $D_{\theta z} = D_{z\theta}$ dispersion terms in (2.5*a*). However, as was shown for solutes by Smith (1990*b*), these additional terms are negligible for rapid rotation except when the total longitudinal dispersion coefficient is dominated by $\kappa_3+D_{zz}$, and so $D_{\theta\theta}$ or $D_{\theta z} = D_{z\theta}$ can be neglected. Here the limiting process of strong buoyancy is different from the limiting process of strong rotation, but the same conclusion holds. To foreshorten the mathematical analysis (and again avoiding replication of the work of Smith 1990*b*), the $D_{\theta\theta}$ and $D_{\theta z} = D_{z\theta}$ and terms have been neglected in (2.5*a*).

Strictly, for polydisperse particles there should be a spectrum of particle diffusivities $\kappa_j$, relative densities $\alpha$, and drift velocity components $V, W$. The total density perturbation (2.4) would then involve integration over the full spectrum. The final outcome would be evolution equations for different parts of the spectrum with nonlinear (buoyant) coupling. The nonlinearity is most dramatic in the limiting case, considered here, of monodisperse particles. For concentrated suspensions a further complication would be that the Brownian diffusivity can be greatly augmented by shear-induced diffusion (Leighton & Acrivos 1987).

## 3. Moving stretched coordinate system

The Taylor (1953) limit concerns the eventual longitudinal dispersion process far up the borehole. In this limit the concentration distribution has become greatly elongated in the axial $z$-direction. For simplicity we shall assume that the gap width $h(\theta)$ varies neither with axial position $z$ nor with time $t$. Thus, to a first approximation the concentration distribution will be carried along at a constant velocity $w_0$. For a solute this velocity is necessarily the cross-sectionally averaged longitudinal velocity $\bar{w}$. However, for rising or sinking particles $w_0$ needs to be determined.

We introduce a small parameter $\delta$ that characterizes the ratio of diameter to axial length in the Taylor regime. In terms of $\delta$ we define a moving stretched coordinate system.

$$\zeta = \delta(z - w_0 t), \quad \tau = \delta^2 t. \tag{3.1a, b}$$

The rescaled version of the advection–diffusion equation (2.5*a*) is

$$\delta^2 h\, \partial_\tau c + \frac{1}{a}\partial_\theta(h[v + V\cos\theta]\, c) + \delta h[w + W - w_0]\, \partial_\zeta c + \delta hc\, \partial_\zeta w$$
$$= \frac{1}{a^2}\partial_\theta(h\kappa_2\, \partial_\theta c) + \delta^2\, \partial_\zeta(h[\kappa_3 + D_{zz}]\, \partial_\zeta c). \tag{3.2}$$

It is the order-$\delta^2$ terms in this equation that are eventually used in determining the slow-time evolution of the concentration distribution.

Since the flow is modified by buoyancy, we have to decide upon the $\delta$-scaling of the gravitational terms in the momentum equations (2.3*a*, *b*). From the $w$-terms in (3.2), we see that it only needs an order-$\delta$ perturbation to $w$ for the slow-time evolution to be changed. Correspondingly, we take the gravitational terms in (2.3*a*, *b*) to be order-$\delta$ relative to the drag terms:

$$\frac{1}{a\delta\rho_0}\partial_\theta p = -12\frac{\nu}{h^2}v + \delta c\alpha g\cos\theta\sin\phi, \tag{3.3a}$$

$$\frac{1}{\rho_0}\partial_\zeta p = -12\frac{\nu}{h^2}w + \delta c\alpha g\cos\phi. \tag{3.3b}$$

The comparatively large scaling for the excess pressure (above reference hydrostatic) is a consequence of the assumed great length of the borehole. We remark that for solutes it requires stronger gravitational terms to influence the shear dispersion process (recall the comparison between (1.2) and (1.3)).

For completeness, we note that in the moving stretched coordinate system the mass conservation equation (2.1) now becomes

$$\frac{1}{a}\partial_\theta(hv)+\delta h\,\partial_\zeta w = 0. \tag{3.4}$$

An important consequence is that the cross-sectionally averaged velocity $\bar{w}$ is constant. For solutes this implies that the Burgers nonlinearity $w_1$ is zero and any buoyancy effects must be evidenced in some other way (Erdogan & Chatwin 1967) and with a different $\delta$-scaling.

## 4. Formal series expansions

To solve (3.2)–(3.4) we pose the regular power series expansions

$$c = c^{(0)}+\delta c^{(1)}+\delta^2 c^{(2)}+\dots,\quad p = p^{(0)}+\delta p^{(1)}+\delta^2 p^{(2)}+\dots, \tag{4.1$a$, $b$}$$

$$v = v^{(0)}+\delta v^{(1)}+\delta^2 v^{(2)}+\dots,\quad w = w^{(0)}+\delta w^{(1)}+\delta^2 w^{(2)}+\dots. \tag{4.1$c$, $d$}$$

We introduce the notation $\langle\dots\rangle$ to denote $\theta$-averages, e.g.

$$\langle h\rangle = \frac{1}{2\pi}\int_0^{2\pi} h\,\mathrm{d}\theta. \tag{4.2}$$

Overbars denote cross-sectional average values, e.g.

$$\bar{c} = \frac{\langle hc\rangle}{\langle h\rangle} = \frac{1}{2\pi\langle h\rangle}\int_0^{2\pi} hc\,\mathrm{d}\theta. \tag{4.3}$$

At leading order the velocity components for laminar constant-viscosity flows are unaffected by buoyancy (see figure 3):

$$v^{(0)} = 0,\quad w^{(0)} = h^2\frac{\langle h\rangle}{\langle h^3\rangle}\bar{w}. \tag{4.4$a$, $b$}$$

We can regard the constant mean axial velocity $\bar{w}$ as being given in terms of the total mass flux of drilling fluid going up the borehole. The corresponding pressure excess (above reference hydrostatic) takes the form

$$p^{(0)} = -12\frac{\langle h\rangle}{\langle h^3\rangle}\rho_0\,\nu\bar{w}(z-z_0)\,\delta,\quad \partial_\theta p^{(1)} = 0. \tag{4.5$a$, $b$}$$

Here the moving coordinate $\zeta$ has been replaced by the stationary axial coordinate $z$. The term $z_0(\tau)$ denotes a reference level which will depend on the detailed flow near the top of the borehole.

We record that the longitudinal component of the shear dispersion tensor has the value

$$D_{zz}^{(0)} = \frac{\bar{w}^2 h^6\langle h\rangle^2}{210\kappa_1\langle h^3\rangle^2}. \tag{4.6}$$

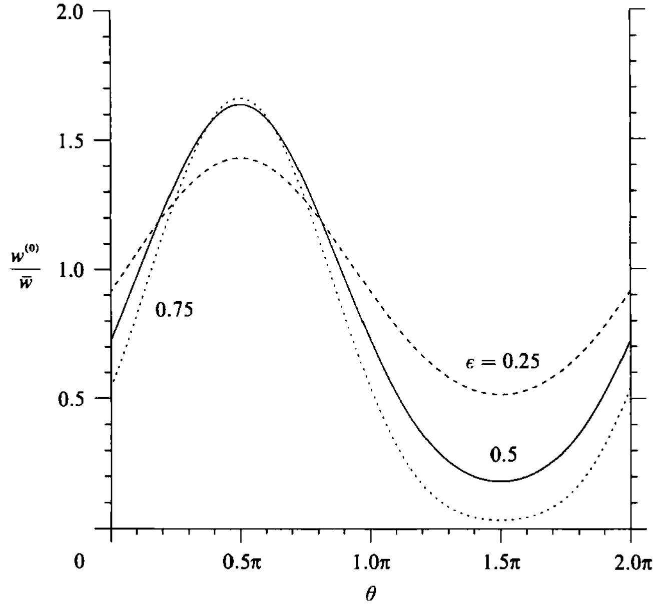


FIGURE 3. Laminar axial velocity for a non-uniform annular gap with narrowest position at the angle $\Theta = \frac{3}{2}\pi$ and with different values of the fractional offset $\epsilon$. If the position of minimum gap width were displaced to another value of $\Theta$, then the velocity profiles would be likewise be displaced.

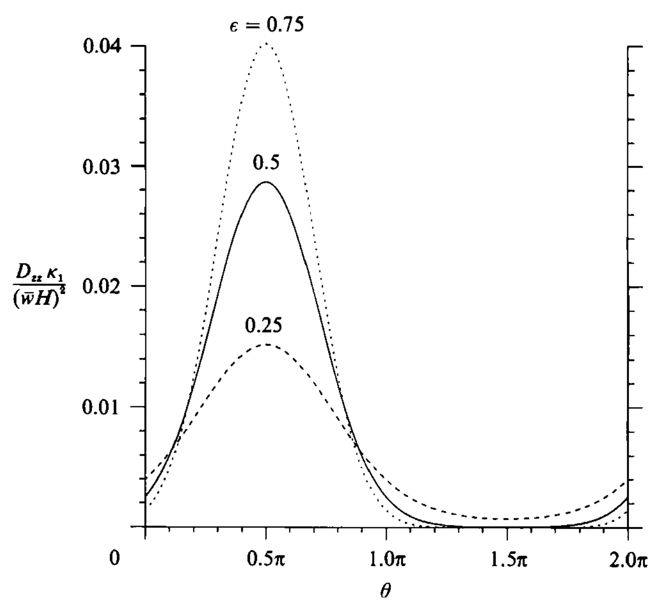


FIGURE 4. Local longitudinal dispersion coefficient $D_{zz}$ for a non-uniform annular gap with narrowest position at the angle $\Theta = \frac{3}{2}\pi$ with different values of the fractional offset $\epsilon$.

Thus, it is regions of wide gap with relatively fast longitudinal flow that exhibit rapid longitudinal spreading in the early stages of the shear dispersion process (see figure 4). At leading order the concentration equation (3.2) has the solution

$$c^0(\theta, \varsigma, \tau) = \bar{c}(\varsigma, \tau)\,\gamma(\theta; V), \tag{4.7a}$$

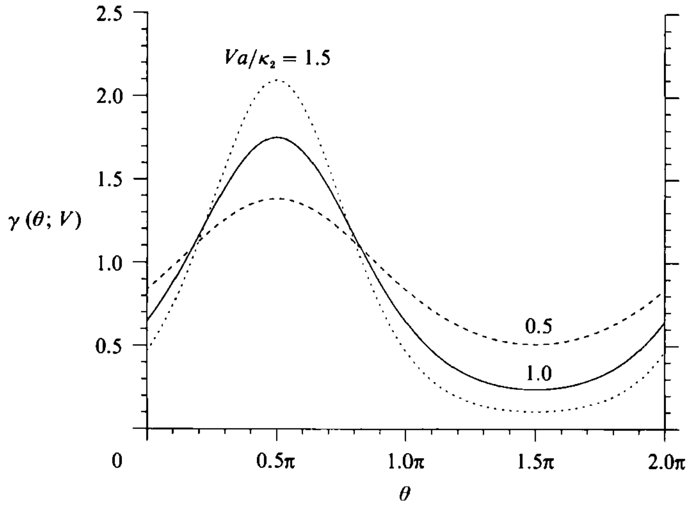


FIGURE 5. Equilibrium profile of monodisperse particles in a non-uniform annular gap with fractional offset $\epsilon = 0.5$, and narrowest gap at the angle $\Theta = \frac{3}{2}\pi$ with different values of the particle rise velocity. For other values of $\Theta, \epsilon$ the vertical scale (but not the shape) is changed (i.e. rising particles tend to be found at the upper side $\theta = \frac{1}{2}\pi$ no matter what the annular geometry).

where
$$\frac{1}{a}\partial_\theta(\gamma V h\cos\theta) = \frac{1}{a^2}\partial_\theta(h\kappa_2\partial_\theta\gamma), \quad \bar{\gamma} = 1. \tag{4.7b, c}$$

The function $\gamma(\theta; V)$ describes the equilibrium distribution of a dilute suspension of particles when there is a balance between the effects of rise and of diffusion. For a laminar flow with constant $\kappa_2$ the solution for $\gamma$ can be written

$$\gamma(\theta; V) = \frac{\langle h\rangle \exp[(aV/\kappa_2)\sin\theta]}{\langle h\exp[(aV/\kappa_2)\sin\theta]\rangle} \tag{4.8}$$

(see figure 5). Essentially $\gamma(\theta; V)$ is a function of vertical position. The precise geometry of the gap (as characterized by the fractional offset $\epsilon$ and the angle $\Theta$ at minimum width) only modifies the normalization factor in the denominator.

As a quantitative illustration of realistic borehole conditions we give the (turbulent) flow specifications:

$$\text{mean radius } a = 100\text{ mm}, \quad \text{mean gap } H = 40\text{ mm}, \tag{4.9a, b}$$

$$\text{bulk velocity } \bar{w} = 100\text{ mm s}^{-1}, \quad \text{particle rise velocity } \mathscr{W} = 5\text{ mm s}^{-1}, \tag{4.9c, d}$$

$$\text{across-gap mixing } \kappa_1 = 25\text{ mm}^2\text{ s}^{-1}, \quad \text{circumferential mixing } \kappa_2 = 100\text{ mm}^2\text{ s}^{-1}, \tag{4.9e, f}$$

$$\text{eddy viscosity } \nu = 25\text{ mm}^2\text{ s}^{-1}. \tag{4.9g}$$

The rise velocity $\mathscr{W}$ corresponds to gas bubbles of radius 0.05 mm with a dirty surface in water. For solid particles of the same size and with a density twice that of water the rise velocity $\mathscr{W}$ would have reversed sign. If the borehole is inclined at an angle $\phi$ of 12° from the vertical, then the rise velocity component $V$ across the borehole has the value

$$V = 1\text{ mm s}^{-1}. \tag{4.9h}$$

it is the magnitude of the combination $aV/\kappa_2$ that determines the non-uniformity of the particle distribution around the shaft. The specification (4.9*a*–*h*) was selected to give round numbers:

$$aV/\kappa_2 = 1. \tag{4.9i}$$

## 5. First-order solutions

The equation for $c^{(1)}$ is

$$\frac{1}{a}\partial_\theta(c^{(1)}Vh\cos\theta)-\frac{1}{a^2}\partial_\theta(h\kappa_2\,\partial_\theta c^{(1)}) = -\partial_\varsigma\bar{c}[w^{(0)}+W-w_0]\gamma h-\frac{\bar{c}}{a}\partial_\theta(h\gamma v^{(1)}). \tag{5.1}$$

A necessary condition for a periodic solution to exist is that the forcing terms have zero $\theta$-average. This condition enables us to evaluate the velocity $w_0$ of the moving frame of reference:

$$w_0 = \overline{\gamma w^{(0)}}+W. \tag{5.2}$$

In any other frame of reference there is not the hypothesized slow-time evolution of $\bar{c}$. The two contributions to $w_0$ are a particle-weighted average of the axial flow $\overline{\gamma w^{(0)}}$ and the axial component $W$ of the rise velocity. If the gap is narrowest at the bottom ($\Theta = \frac{3}{2}\pi$), as illustrated in figure 1, the buoyant particles can move along the borehole much faster than the mean flow velocity $\bar{w}$. Indeed, for the flow specification (4.9*a*–*i*) we can estimate that $w_0 = 130$ mm s$^{-1}$ whereas $\bar{w} = 100$ mm s$^{-1}$. Conversely, if the gap is narrowest at the top ($\Theta = \frac{1}{2}\pi$), the buoyant particles move relatively slowly (see figure 6), and we obtain the estimate $w_0 = 75$ mm s$^{-1}$.

The buoyancy terms in the momentum equations (3.3*a*, *b*) give rise to the first-order velocity corrections

$$v^{(1)} = \bar{v}^{(1)}\frac{\langle h\rangle}{h} \quad\text{with}\quad \bar{v}^{(1)} = -\frac{\bar{c}\alpha g\sin\phi}{12\nu\langle h\rangle\langle h^{-3}\rangle}\langle\sin\theta\,\partial_\theta\gamma\rangle = 0, \tag{5.3a, b}$$

$$w^{(1)} = \bar{c}\frac{\alpha g\cos\phi h^2}{12\nu}\left(\gamma-\frac{\langle\gamma h^3\rangle}{\langle h^3\rangle}\right), \tag{5.3c}$$

$$\partial_\zeta p^{(1)} = \rho_0\,\bar{c}\alpha g\cos\phi\frac{\langle\gamma h^3\rangle}{\langle h^3\rangle}. \tag{5.3d}$$

The buoyancy-driven circumferential velocity $v^{(1)}$ is related to any asymmetry in the particle distribution around the annular gap. Such asymmetry is absent when there is no rotation (see (4.8)). For the axial buoyancy-driven flow $w^{(1)}$ there is an upwards contribution where the normalized particle fraction $\gamma(\theta)$ of buoyant particles exceeds the flux-weighted average $\langle\gamma h^3\rangle/\langle h^3\rangle$, with a downwards contribution elsewhere. The buoyancy correction $\partial_\zeta p^{(1)}$ to the pressure gradient ensures that $w^{(1)}$ does not modify the total mass flux of drilling fluid going up the borehole.

Since $v^{(1)}$ is zero, we can infer from (5.1) that the perturbation concentration $c^{(1)}$ can be represented

$$c^{(1)} = -\gamma f\partial_\varsigma\bar{c}, \tag{5.4}$$

where the centroid displacement function $f(\theta; V)$ satisfies the equation

$$-\frac{1}{a^2}\partial_\theta(h\gamma\kappa_2\,\partial_\theta f) = [w^{(0)}-\overline{\gamma w^{(0)}}]\gamma h \tag{5.5a}$$

with

$$\overline{\gamma f} = 0. \tag{5.5b}$$

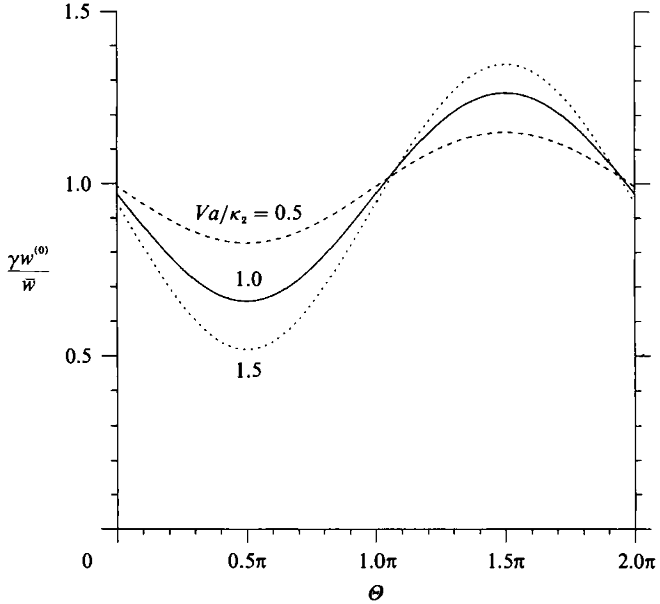


FIGURE 6. Particle-weighted longitudinal velocity $\overline{\gamma w^{(0)}}$ of monodisperse particles in a non-uniform annular gap with fractional offset $\epsilon = 0.5$ for different values of the particle rise velocity $V$. If the narrowest gap coincides with where the particles accumulate ($\theta = \frac{1}{2}\pi$) then the effective velocity has a minimum.

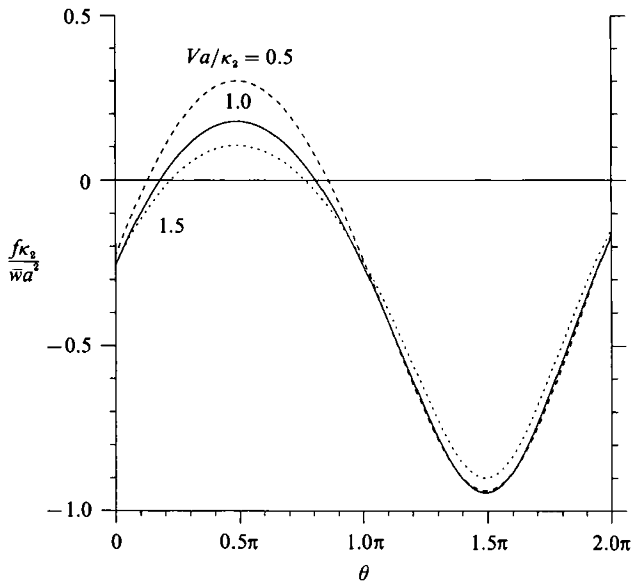


FIGURE 7. Centroid displacement $f$ of monodisperse particles in a non-uniform annular gap with fractional offset $\epsilon = 0.5$, and narrowest gap at the angle $\Theta = \frac{3}{2}\pi$, for different values of rise velocity $V$.

If the gap width $h(\theta)$ is non-uniform, then there is an upwards displacement of the concentration distribution where the gap is widest and the axial flow is fastest. To satisfy the normalization (5.5*b*) there is a compensating downwards displacement where the gap is narrow and the flow is slow (see figure 7). For the flow specification (4.9*a*–*i*) the centroid displacement $f$ has magnitude $\bar{w}a^2/\kappa_2 = 10$ m.

For laminar flow with $\kappa_2$ constant, a first integral of equation (5.5*a*) is

$$\frac{h\gamma}{a^2}\kappa_2\,\partial_\theta f = J(\theta;V)\,\bar{w}\langle h\rangle, \tag{5.6a}$$

with

$$\bar{w}\langle h\rangle\,J(\theta;V) = -\int_0^\theta [w^{(0)}-\overline{\gamma w^{(0)}}]\,\gamma h\,\mathrm{d}\theta' + \frac{1}{\langle h^{-1}\gamma^{-1}\rangle}\left\langle \frac{1}{h\gamma}\int_0^\theta [w^{(0)}-\overline{\gamma w^{(0)}}]\,\gamma h\,\mathrm{d}\theta'\right\rangle. \tag{5.6b}$$

The $\theta$-averaged constant term in the definition (5.6*b*) of $J$ ensures that the integral $\partial_\theta f$ around the gap is zero. To perform the second integration it is convenient for us to introduce a partition of unity

$$u_-(\theta) = \frac{1}{2\pi\langle h\gamma\rangle}\int_0^\theta h\gamma\,\mathrm{d}\theta', \quad u_+(\theta) = \frac{1}{2\pi\langle h\gamma\rangle}\int_\theta^{2\pi} h\gamma\,\mathrm{d}\theta', \tag{5.7a, b}$$

$$u_- + u_+ = 1. \tag{5.7c}$$

The weight factor $h\gamma$ in the integrals (5.7 *a*, *b*) is related to the required normalization (5.5*b*) of $f$. The solution for $f(\theta;V)$ is

$$\frac{\kappa_2 f}{a^2} = \bar{w}\int_0^\theta \langle h\rangle\frac{u_-}{h\gamma}J(\theta';V)\,\mathrm{d}\theta' - \bar{w}\int_\theta^{2\pi}\langle h\rangle\frac{u_+}{h\gamma}J(\theta';V)\,\mathrm{d}\theta'. \tag{5.8}$$

## 6. Longitudinal dispersion equation

The $\theta$-average of the rescaled advection–diffusion equation (3.2) is

$$\delta^2\,\partial_\tau\langle hc\rangle + \delta\,\partial_\varsigma\langle h[w+W-w_0]\,c\rangle = \delta^2\,\partial_\zeta\langle h[\kappa_3+D_{zz}]\,\partial_\varsigma c\rangle. \tag{6.1}$$

The selection of the velocity $w_0$ for the moving frame of reference (5.2), removes the terms of order $\delta$. The terms of order $\delta^2$ yield an evolution equation for $\bar{c}(\varsigma,\tau)$:

$$\langle h\rangle\,\partial_\tau\bar{c} + \partial_\varsigma(\langle h\gamma w^{(1)}\rangle\,\bar{c}) + \partial_\varsigma\langle h[w^{(0)}-\overline{\gamma w^{(0)}}]\,c^{(1)}\rangle = \langle h\gamma[\kappa_3+D^{(0)}_{zz}]\rangle\,\partial^2_\varsigma\bar{c}. \tag{6.2}$$

If we use (5.3*c*) for $w^{(1)}$ and (5.4) for $c^{(1)}$, then this evolution equation (6.2) for $\bar{c}$ can be written as a Burgers (1948) equation:

$$\partial_\tau\bar{c} + w_1\,\bar{c}\,\partial_\varsigma\bar{c} = [\overline{\gamma\kappa_3} + \overline{\gamma D^{(0)}_{zz}} + E]\,\partial^2_\varsigma\bar{c}, \tag{6.3a}$$

with

$$w_1 = \frac{\alpha g\cos\phi}{6\nu\langle h\rangle}\left\langle h^3\left(\gamma - \frac{\langle\gamma h^3\rangle}{\langle h^3\rangle}\right)^2\right\rangle, \tag{6.3b}$$

$$E = \frac{1}{\langle h\rangle}\langle\gamma h[w^{(0)}-\overline{gw^{(0)}}]f\rangle = \frac{1}{a^2\langle h\rangle}\langle\kappa_2\,\gamma h(\partial_\theta f)^2\rangle. \tag{6.3c}$$

The sign of the nonlinear velocity coefficient $w_1$ is positive for buoyant particles (with $\alpha$ positive) and negative for dense particles (with $\alpha$ negative). The magnitude of $w_1$ is related principally to the non-uniformity of $\gamma$. The nonlinearity is marginally larger when the gap is widest at the side (see figure 8). We recall that for particles

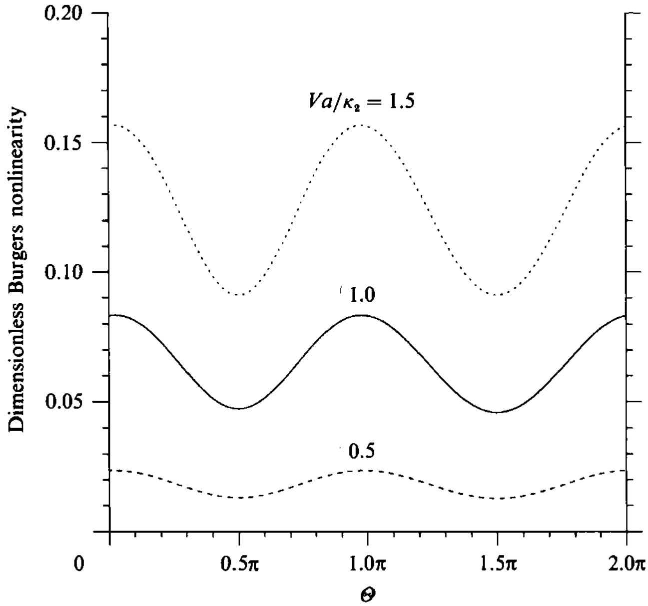


FIGURE 8. Dimensionless Burgers nonlinearity $w_1 \nu/\alpha g H^2 \cos\phi$ for monodisperse particles in a non-uniform annular gap with fractional offset $\epsilon = 0.5$ for different values of the rise velocity $V$.

with a given vertical rise velocity $\mathscr{W}$, the transverse component $V$ as defined in (2.5*b*) is proportional to $\sin\phi$. Hence the nonlinearity $w_1$ vanishes if either the borehole is horizontal ($\cos\phi = 0$) or vertical ($\sin\phi = 0$). It is only in tilted boreholes that buoyant particles satisfy a Burgers equation.

For gas bubbles with $\alpha g = 10^4$ mm s$^{-2}$ and with the flow specification (4.9*a*–*i*) the dimensional factor in the formula (6.3*b*) for $w_1$ can be estimated:

$$\frac{\alpha g H^2 \cos\phi}{\nu} = 6.4 \times 10^5 \text{ mm s}^{-1}. \tag{6.4}$$

Thus, a dimensionless Burgers nonlinearity of magnitude 0.05 in figure 8 corresponds to a dimensional velocity correction

$$3.2 \times 10^4 \bar{c} \text{ mm s}^{-1}. \tag{6.5}$$

For example, the buoyancy effect associated with a gas volume fraction $\bar{c}$ of only one part per thousand augments the effective gas speed up the borehole by 32 mm s$^{-1}$.

The first integral (5.6*a*) for $\partial_\theta f$ enables us to represent the shear dispersion coefficient $E$ associated with velocity differences around the gap by $\theta$-integrals:

$$E = \frac{a^2 \bar{w}^2}{\kappa_2} \hat{E}(V, \epsilon, \Theta), \tag{6.6a}$$

where
$$\hat{E} = \langle \langle h \rangle J^2 / \gamma h \rangle. \tag{6.6b}$$

The particle-weighted velocity differences are reduced by the upwards drift. So the dimensionless shear dispersion coefficient $\hat{E}$ tends to be decreased as $V$ increases (see figure 9). The scaling for the shear dispersion associated with velocity differences across the gap is

$$\overline{\gamma D_{zz}^{(0)}} = \frac{\langle h \rangle^2 \bar{w}^2}{\kappa_1} \hat{D}(V, \epsilon, \Theta), \tag{6.7}$$

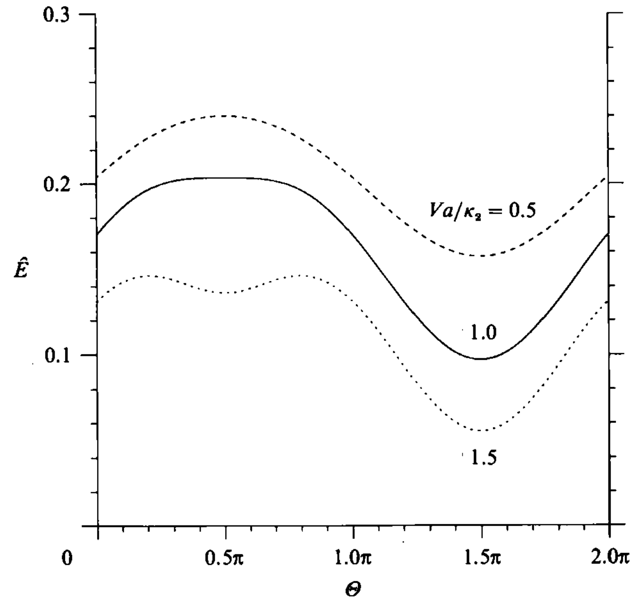


FIGURE 9. Dimensionless longitudinal shear dispersion coefficient $\hat{E}$ associated with velocity differences around the non-uniform gap, for fractional offset $\epsilon = 0.5$ with different values of the particle rise velocity $V$.

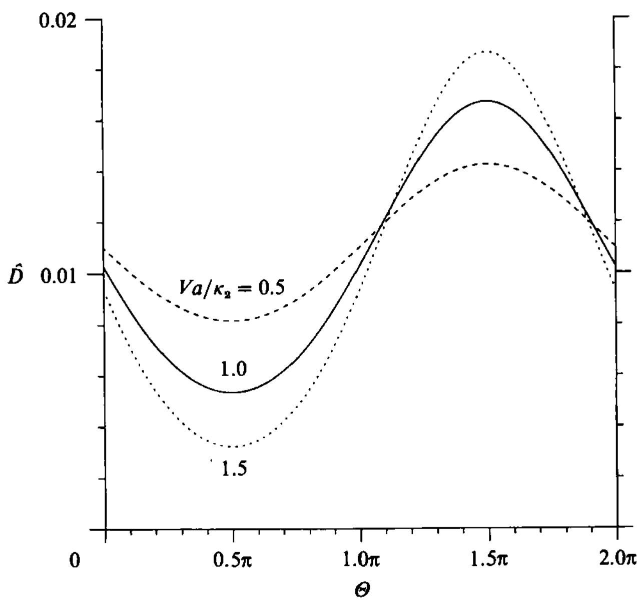


FIGURE 10. Dimensional longitudinal shear dispersion coefficient $\hat{D}$ associated with velocity differences across the gap, for fractional offset $\epsilon = 0.5$ with different values of the particle rise velocity $V$.

where $\hat{D}$ is dimensionless (see figure 10). Not only is $\hat{D}$ numerically smaller than $\hat{E}$, but also the scaling of $\overline{\gamma D_{zz}^{(0)}}$ is smaller by a factor $(\langle h \rangle / a)^2$ than the scaling of $E$. However, unlike $\hat{E}$, $\hat{D}$ does not tend to zero as $\epsilon$ tends to zero or as $aV/\kappa_2$ increases. So, the longitudinal shear dispersion is dominated by velocity differences around the gap unless either the drilling shaft is exactly centralized ($\epsilon = 0$) or $aV/\kappa_2$ is very large.

For the flow specification (4.9*a*–*i*) and with $\Theta = \frac{3}{2}\pi$, a dimensionless dispersion coefficient $\hat{E} = 0.1$ in figure 9 corresponds to a dimensional shear dispersion coefficient

$$E = 10^5 \text{ mm}^2 \text{ s}^{-1} = 0.1 \text{ m}^2 \text{ s}^{-1}. \tag{6.8}$$

Similarly, a dimensionless dispersion coefficient $\hat{D} = 0.016$ in figure 10 corresponds to a dimensional shear dispersion coefficient

$$D = 10^4 \text{ mm}^2 \text{ s}^{-1} = 0.01 \text{ m}^2 \text{ s}^{-1}. \tag{6.9}$$

$E$ would be larger and $D$ smaller if the minimum gap position $\Theta$ were displaced away from the bottom. We remark that on the timescale $10^4$ s for the drilling fluid to rise 1 km the diffusive spread is about $4.5 \times 10^4$ mm = 45 m.

## 7. Solutions of the Burgers equation

If we combine $\overline{\gamma\kappa_3} + \overline{\gamma D_{zz}^{(0)}} + E$ into a total dispersion coefficient $D_0$, then the longitudinal dispersion equation (6.3*a*) is written

$$\partial_\tau \bar{c} + w_1 \bar{c}\, \partial_\varsigma \bar{c} = D_0 \partial_\varsigma^2 \bar{c}. \tag{7.1}$$

Back in the stationary unstretched axes, this equation becomes

$$\partial_t \bar{c} + (w_0 + \delta w_1 \bar{c})\, \partial_z \bar{c} = D_0 \partial_z^2 \bar{c}. \tag{7.2}$$

The factor $\delta$ stems from the formal assumption that the ratio of diameter to axial length is small (of the order $\delta$). In the remainder of this section the smallness of the nonlinear velocity correction will be accounted for via (5.3*c*) for $w_1$ and any $\delta$ factors will be suppressed.

For small particle concentrations the evolution of $\bar{c}(z, t)$ will be the same as that for a linear diffusion equation. Hopf (1950) and Cole (1951) showed that even when $\bar{c}$ is not small, the nonlinearity can be eliminated. We define the concentration $b$:

$$b(z,t) = \bar{c} \exp\left(\frac{w_1}{2D_0}\int_z^\infty \bar{c}(z', t)\, \mathrm{d}z'\right). \tag{7.3}$$

The integral implies that we are assuming that $\bar{c}$ tends to zero as $z$ tends to plus infinity. Remarkably, the evolution of $b$ is linear diffusive:

$$\partial_t b + w_0 \partial_z b = D_0 \partial_z^2 b. \tag{7.4}$$

So, it is much easier to find a solution (numerical or analytical) for $b$ than for $\bar{c}$. To construct $\bar{c}$ we need to invert the transformation (7.3):

$$\bar{c}(z,t) = b\left\{1 + \frac{w_1}{2D_0}\int_z^\infty b(z', t)\, \mathrm{d}z'\right\}^{-1}. \tag{7.5}$$

The effect of the Burgers equation nonlinearity is acounted for in the quotient nonlinearity of this inversion formula (7.5).

A classical example (Taylor 1910) which shows the possibility of a balance between nonlinearity and diffusion is

$$b(z,t) = \xi \exp\left(-[z - (w_0 + \tfrac{1}{2}\xi w_1) t]\, \xi w_1 / 2D_0\right), \tag{7.6a}$$

$$\bar{c}(z,t) = \tfrac{1}{2}\xi\{1 - \tanh\left([z - (w_0 + \tfrac{1}{2}\xi w_1) t]\, \xi w_1 / 4D_0\right)\}. \tag{7.6b}$$

The positive constant, $\xi$, characterizes both the amplitude and the abruptness of the concentration surge of buoyant particles. (In a blowout this tendency for a surge to hold together is compounded by expansion effects.) For sinking particles the sign of the nonlinearity is reversed. With $w_1$ negative the exact solution (7.6*b*) describes the flushing out of the borehole of an abrupt absence of particles.

If we use the flow specification (4.9*a–i*) with

$$\xi = 10^{-3}, \quad \Theta = \tfrac{3}{2}\pi, \tag{7.7a, b}$$

then the lengthscale of the concentration surge is

$$2D_0/(\xi w_1) = 6250 \text{ mm} = 6.25 \text{ m}. \tag{7.8}$$

This is smaller than the diffusive spread estimated in the previous section. So the surge structure would be established in less than 1 km. The speed of the surge is

$$w_0 + \tfrac{1}{2}\xi w_1 = 146 \text{ mm s}^{-1} \tag{7.9}$$

and is markedly greater than the bulk velocity $\bar{w} = 100$ mm s$^{-1}$.

## 8. Concluding remarks

Although this paper focuses upon a particular flow geometry of direct concern to the oil drilling industry, much of the mathematical analysis and physical mechanisms are of more general applicability. The two essential ingredients for a Burgers concentration dispersion equation are particle drift across the flow and a longitudinal force on the flow proportional to the number density of particles. Here it is gravity that causes both the drift and the force. It is easy to envisage other tilted flow geometries where gravity will have the same two effects. Similarly, it is easy to envisage other mechanisms (centrifugal, chemical, electrical, magnetic, thermal) that can provide either the transverse drift or the longitudinal force or both (Lightfoot, Chiang & Noble 1981).

REFERENCES


Bugliarello, G. & Jackson, E. D. 1964 Random walk study of convective diffusion. *J. Engng Mech. Div. ASCE* **90**, 49–77.

Burgers, J. M. 1948 A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* **1**, 171–199.

Cole, J. D. 1951 On a quasi-linear parabolic equation occurring in aerodynamics. *Q. Appl. Maths* **9**, 225–236.

Elder, J. W. 1959 The dispersion of marked fluid in turbulent shear flow. *J. Fluid Mech.* **5**, 544–560.

Erdogan, M. E. & Chatwin, P. C. 1967 The effects of curvature and buoyancy on the laminar dispersion of solute in a horizontal tube. *J. Fluid Mech.* **29**, 465–484.

Giddings, J. C. 1968 Non-equilibrium theory of field-flow fractionation. *J. Chem. Phys.* **49**, 81–85.

Herbolzheimer, E. & Acrivos, A. 1981 Enhanced sedimentation in narrow tilted channels. *J. Fluid Mech.* **108**, 485–499.

Hopf, E. 1950 The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Commun. Pure Appl. Maths* **3**, 201–230.

Leighton, D. & Acrivos, A. 1987 The shear-induced migration of particles in concentrated suspensions. *J. Fluid Mech.* **181**, 415–439.

Lightfoot, E. N., Chiang, A. S. & Noble, P. T. 1981 Field-flow fractionation (polarisation chromatography). *Ann. Rev. Fluid Mech.* **13**, 351–378.

Muller, S. J. & Bittleston, S. H. 1992 Taylor & Aris dispersion of generalised Newtonian fluid for axial flow in an eccentric annulus (in preparation).

Sankarasubramanian, R. & Gill, W. N. 1971 Taylor diffusion in laminar flow in an eccentric annulus. *Intl J. Heat Mass Transfer* **14**, 905–919.

Schlichting, H. 1955 *Boundary Layer Theory*. Pergamon.

Smith, R. 1990*a* Two-dimensional shear dispersion for skewed flows in narrow gaps between moving surfaces. *J. Fluid Mech.* **214**, 211–228.

Smith, R. 1990*b* Shear dispersion along an axle in a closely fitting shaft. *J. Fluid Mech.* **219**, 647–658.

Snyder, W. T. & Goldstein, G. A. 1965 An analysis of fully developed laminar flow in an eccentric annulus. *AIChE J.* **11**, 462–467.

Taylor, G. I. 1910 The conditions necessary for discontinuous motion in gases. *Proc. R. Soc. Lond.* A **84**, 371–377.

Taylor, G. I. 1953 Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. R. Soc. Lond.* A **216**, 186–203.